

§12. Topological Spaces.

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Def. A topology on a set X is a collection $\mathcal{T}$ of subsets of X s.t.

(1) $\emptyset, X \in \mathcal{T}$

(2) The union of the elements in any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Elements of $\mathcal{T}$ are called open sets.

ex. (1) The discrete topology 2^X

(2) The indiscrete topology $\{\emptyset, X\}$

(3) Finite complement topology $\{\emptyset\} \cup \{X \setminus F : F \text{ finite}\}$.

(4) Countable complement topology $\{\emptyset\} \cup \{X \setminus C : C \text{ countable}\}$.

Def $\mathcal{T}, \mathcal{T}'$ topologies on X .

$\mathcal{T}'$ is finer than $\mathcal{T}$, $\mathcal{T}$ is coarser than $\mathcal{T}'$, if $\mathcal{T}' \supset \mathcal{T}$.

§13. Basis for a topology.

Def X is a set, a basis for a topology on X is a collection $\mathcal{B} \subseteq 2^X$ s.t.

(1) $\forall x \in X \exists B \in \mathcal{B} \ x \in B$

(2) $x \in B_1 \cap B_2, B_1, B_2 \in \mathcal{B} \rightarrow \exists B_3 \in \mathcal{B} \ x \in B_3 \subseteq B_1 \cap B_2$.

A basis $\mathcal{B}$ generates a topology where U is open if $\forall x \in U \exists B \in \mathcal{B} \ x \in B \subseteq U$.

ex. (1) discs, rectangles form a basis for the ordinary topology on $\mathbb{R}^2$.

(2) the singletons form a basis for the discrete topology.

Lemma. $(X, \mathcal{T})$ topology, $\mathcal{C}$ ^{collection of} open sets s.t. $\forall x \in U \text{ open} \rightarrow \exists C \in \mathcal{C} \ x \in C \subseteq U$, then $\mathcal{C}$ is a basis for $\mathcal{T}$.

Lemma. $\mathcal{B}$ basis for $\mathcal{T}$, $\mathcal{B}'$ basis for $\mathcal{T}'$.

$$\mathcal{T}' \text{ finer than } \mathcal{T} \iff \forall x \in X \forall B \in \mathcal{B} \exists B' \in \mathcal{B}' \ x \in B' \subseteq B.$$

ex. $\mathbb{R}$, ordinary topology has basis (a, b) , $a < b$.

$\mathbb{R}_\ell$, "lower-limit topology" has basis $[a, b)$, $a < b$.

$\mathbb{R}_K$, "K-topology" has basis (a, b) , $(a, b) \setminus K$ where $K = \{1/n : n \in \mathbb{Z}_{>0}\}$.

$\Rightarrow \mathbb{R}_\ell, \mathbb{R}_K$ finer than $\mathbb{R}$.

Def. A subbasis for a topology of X is a subcollection $\mathcal{S}$ s.t. $\bigcup \mathcal{S} = X$.

Subbasis $\xRightarrow{\text{finite intersections}}$ basis $\xRightarrow{\text{unions}}$ topology

§14. The Order Topology.

Def. A simple order $<$ on a set X is a relation s.t.

(1) (Comparability) $x \neq y \rightarrow x < y$ or $y < x$

(2) (Nonreflexivity) $\forall x \in X \quad \neg(x < x)$

(3) (Transitivity) $x < y$ and $y < z \rightarrow x < z$.

Def. Let X be a simply ordered set. The sets
(with at least two elts)

$$\{x \in X: x < a\}, \{x \in X: x > a\}, \quad a \in X$$

form a subbasis for a topology on X , called the order topology.

ex) (1) standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$.

(2) Given two ordered sets X, Y , may give $X \times Y$ the lexicographic order: $(x_1, y_1) < (x_2, y_2) \leftrightarrow x_1 < x_2$ or $(x_1 = x_2 \text{ and } y_1 < y_2)$.

I_0^2 , the ordered square, is $[0, 1]^2$ w/ order topology from lexicographic order.

(3) Well-ordered sets (see §10).

§15. The Product Topology.

Def. X, Y top'l sps, $\pi_1: X \times Y \rightarrow X, (x, y) \mapsto x$, $\pi_2: X \times Y \rightarrow Y, (x, y) \mapsto y$.

product topology on $X \times Y$

= topology w/ basis $U \times V$, $U \subseteq X, V \subseteq Y$ open

= topology w/ subbasis $\pi_1^{-1}(U), \pi_2^{-1}(V)$, $U \subseteq X, V \subseteq Y$ open

= topology w/ basis $B \times C$, $B \in \mathcal{B}, C \in \mathcal{C}$, $\mathcal{B}$ basis for X , $\mathcal{C}$ basis for Y .

§16. The subspace topology.

Def X topol sp, $Y \subseteq X$. Then

$$\{Y \cap U : U \subseteq X \text{ open}\}$$

is a topology on Y , called the subspace topology

Lemma B basis for X , $Y \subseteq X$, then $\{B \cap Y : B \in B\}$ basis for subsp top of Y .

Lemma Y subsp of X , U open in Y , Y open in $X \rightarrow U$ open in X .

Exercise subspace top. of product top. is product top. of subsp top.

§17. Closed Sets and Limit Points.

Def. $A \subseteq X$ is closed if $X \setminus A$ is open.

Thm. X top'l space.

(a) $\emptyset, X$ closed.

(b) Arbitrary intersections of closed sets are closed.

(c) Finite unions of closed sets are closed.

Specifying a topology (the open sets) is equiv. to specifying the closed sets.

• Under subspace topologies:

Thm. Y subspace of X . A closed in Y iff $A = Y \cap F$ for F clsd in X .

Thm. A closed in Y , Y closed in X (Y subsp of X) $\Rightarrow A$ clsd in X .

• Def. $A \subseteq X$.

• interior of A , $\text{int } A$ = largest open set contained in A
 = union of ^{all} open sets contained in A

• closure of A , $\bar{A}$ = smallest closed set containing A
 = intersection of all closed sets containing A

Thm. $A \subseteq Y \subseteq X$. (closure of A in Y) = $\bar{A} \cap Y$.

How to find $\bar{A}$.

Thm. $A \subseteq X$.

$x \in \bar{A} \iff$ every nbhd of x (an open set containing x) intersects A

$\iff$ every basis nbhd of x intersects A .

• Def x is a limit pt of $A \subseteq X$ if $x \in \overline{A \setminus \{x\}}$.
 i.e. every nbhd of x intersects $A \setminus \{x\}$.

Thm $\bar{A} = A \cup A'$
 $\uparrow$ limit pts of A .

Cor A closed iff $A' \subseteq A$.

Def X is Hausdorff if $\forall x_1 \neq x_2$ in X $\exists$ nbhd U_1 of x_1 , U_2 of x_2
 $U_1 \cap U_2 = \emptyset$.

Def X is T_1 if every pt in X is closed.

Thm $Hdf \Rightarrow T_1$.

Thm $X T_1$, $A \subseteq X$. x is a limit pt of A iff $\forall$ nbhd of x
has ∞ -many pts of A .

Def $\{x_n\} \rightarrow x$ if

Thm Limits are unique in Hdf spaces.

Thm Order top. is Hdf.

$Hdf \times Hdf$ is Hdf. Subsp of Hdf is Hdf.

§18. Continuous Functions.

X, Y : topl sp.

Def A function $f: X \rightarrow Y$ is continuous at x if $\forall$ nbhd V of $f(x)$
 $\exists$ nbhd U of x s.t. $f(U) \subseteq V$.

A function $f: X \rightarrow Y$ is continuous if $\forall$ open $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open.

Thm X, Y topl sps, $f: X \rightarrow Y$ function. TFAE.

(1) f conti.

(2) $\forall A \subseteq X$ $f(A) \subseteq \overline{f(A)}$

(3) $\forall B \subseteq Y$ closed, $f^{-1}(B) \subseteq X$ closed.

(4) f conti at every pt of $x \in X$.

• Homeomorphisms.

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Defn. X, Y topol sp. A map $f: X \rightarrow Y$ is a homeomorphism if it is bijective and f, f^{-1} are continuous.

Properties of topological spaces that are invariant under homeomorphisms are called topological properties.

Defn. A continuous injection $f: X \rightarrow Y$ is a topological imbedding if the map $f: X \rightarrow f(X)$, with $f(X)$ given the subspace topology from Y , is a homeomorphism.

• Constructing continuous functions.

Thm

- $f: X \rightarrow Y$, where $f(x) = y_0 \forall x$ for a fixed $y_0 \in Y$, is conti.
- $A \subseteq X$, inclusion map $j: A \rightarrow X$ is conti.
- $f: X \rightarrow Y, g: Y \rightarrow Z$ conti $\Rightarrow g \circ f: X \rightarrow Z$ conti.
- $f: X \rightarrow Y$ conti, $A \subseteq X \Rightarrow f|_A: A \rightarrow Y$ conti.
- $f: X \rightarrow Y$ conti, $f(X) \in Z \subseteq Y \subseteq W$, then restriction $g: X \rightarrow Z$, extension $h: X \rightarrow W$ conti.
- If $f: X \rightarrow Y$ map s.t. $X = \bigcup U_\alpha, U_\alpha$ open, $f|_{U_\alpha}: U_\alpha \rightarrow Y$ conti, then f conti.

Pasting lemma.

$X = A \cup B$, A, B closed, $f: A \rightarrow Y, g: B \rightarrow Y$ conti s.t. $f|_{A \cap B} = g|_{A \cap B}$, then $h: X \rightarrow Y, h = \begin{cases} f & \text{on } A \\ g & \text{on } B \end{cases}$ is conti.

Continuity of product map.

$f: X \rightarrow Y \times Z$ conti iff $\pi_1 \circ f: X \rightarrow Y, \pi_2 \circ f: X \rightarrow Z$ conti.
(where $\pi_1: Y \times Z \rightarrow Y, \pi_2: Y \times Z \rightarrow Z$ proj. maps.)

Algebra property of $\mathbb{R}$ -valued conti maps.

~~Real~~ Real-valued conti maps closed under addition, scalar multiplication, products, quotients (by nonzero maps), and uniform limits.

§19. The Product Topology

Notation. Given an index set J and sets $A_\alpha, \alpha \in J$, the Cartesian product $\prod_{\alpha \in J} A_\alpha$ consists of J -tuples $(x_\alpha)_{\alpha \in J}$, where each $x_\alpha \in A_\alpha$.

The Axiom of Choice states that if each $A_\alpha \neq \emptyset$, then $\prod_{\alpha \in J} A_\alpha \neq \emptyset$.

Defn. $\{X_\alpha\}_{\alpha \in J}$ top'l spaces.

- Projection maps $\pi_\alpha: \prod_{\beta \in J} X_\beta \rightarrow X_\alpha, (x_\beta)_{\beta \in J} \mapsto x_\alpha$.
- The box topology on $\prod_{\beta \in J} X_\beta$ is the topology with basis $\{\prod_{\beta \in J} U_\beta: U_\beta \text{ open in } X_\beta\}$.
- The product topology on $\prod_{\beta \in J} X_\beta$ is the topology with subbasis $\{\pi_\beta^{-1}(U_\beta): U_\beta \text{ open in } X_\beta, \beta \in J\}$ or with basis $\{\prod_{\beta \in J} U_\beta: U_\beta \text{ open in } X_\beta, U_\beta = X_\beta \text{ for all but finitely many } \beta\}$.

(May replace U_β by B_β where B_β range over a basis of X_β .)

Thm.

- $A_\alpha \subset X_\alpha$ subsp.

box/product topology of $\prod_\alpha A_\alpha = \text{subsp topology of } \prod_\alpha A_\alpha \text{ from } \prod_\alpha X_\alpha \text{ w/ box/product topology.}$

- If each X_α Hdf, then $\prod_\alpha X_\alpha$ w/ box/prod. top is Hdf.

- $A_\alpha \subset X_\alpha, \overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}$ in box/prod. top.

The following suggests that the product topology is more "natural".

Thm.

A function $f: A \rightarrow \prod_\alpha X_\alpha$ is conti iff $\pi_\alpha \circ f$ is conti $\forall \alpha$.
 $\downarrow$ w/ prod. top.

The Tychonoff thm, which we will learn later in the course, states that the product of "compact" top'l spaces w/ the product top is "compact".

§20. The Metric Topology.

Def. A metric on a set X is a function $d: X \times X \rightarrow \mathbb{R}$ s.t.

- (1) (nonnegativity). $d(x, y) \geq 0 \quad \forall x, y \in X. \quad d(x, y) = 0 \iff x = y.$
- (2) (symmetry) $d(x, y) = d(y, x) \quad \forall x, y \in X$
- (3) (triangle ineq). $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X.$

The balls

$$B(x, r) = \{y \in X: d(x, y) < r\}$$

form a basis for a topology on X , called the metric topology.

A topol sp X is metrizable if it is the metric topology of some metric d .

A metric space is a set X w/ a metric d along w/ its metric topology.

Rmk. Metric properties are not topol properties.

e.g. a subset A is bounded if its diameter

$$\text{diam } A = \sup_{a, b \in A} d(a, b)$$

is finite. But any metric d can be replaced by $\min\{d, 1\}$.

Lemma. Let d, d' be two metrics on a set X , let τ, τ' be the topologies they induce, respectively. Then τ' is finer than τ if and only if $\forall x \in X \forall \varepsilon > 0 \exists \delta > 0$

$$B_{d'}(x, \delta) \subset B_d(x, \varepsilon).$$

Cor. Suppose two metrics d, d' on a set X are comparable: $\exists a, b > 0$ s.t.

$$a d(x, y) \leq d'(x, y) \leq b d(x, y), \quad \forall x, y \in X.$$

Then d and d' induce the same topology on X .

Ex. Consider the Euclidean (l_2) metric on $\mathbb{R}^d$:

$$d_2(\vec{x}, \vec{y}) = \left(\sum_{i=1}^d |x_i - y_i|^2 \right)^{1/2},$$

the square (l_∞) metric on $\mathbb{R}^d$:

$$d_\infty(\vec{x}, \vec{y}) = \max_{1 \leq i \leq d} |x_i - y_i|$$

and more generally the l_p ($1 \leq p < \infty$) metric on $\mathbb{R}^d$:

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^d |x_i - y_i|^p \right)^{1/p}.$$

These are indeed metrics on $\mathbb{R}^d$ and are all comparable:

$$d_\infty(x, y) \leq d_p(x, y) \leq d^{1/p} d(x, y).$$

So they all induce the same topology on $\mathbb{R}^d$.

$\mathbb{R}^d$ with the l_p metric is also denoted l_p^d .

Optional Note.

More generally, a norm on a vector space X is a function $\|\cdot\|: X \rightarrow \mathbb{R}$ s.t.

$$1) \|x\| \geq 0 \quad \forall x \in X, \text{ and } \|x\| = 0 \iff x = 0.$$

$$2) \|ax\| = |a| \|x\| \quad \forall a \in \mathbb{R} \quad \forall x \in X$$

$$3) \|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X.$$

A normed space is a vector space X equipped with a norm $\|\cdot\|$.

Given a normed space $(X, \|\cdot\|)$, the function $d: X \times X \rightarrow \mathbb{R}$, $d(x, y) = \|x - y\|$ is a distance, and we automatically assume that $(X, \|\cdot\|)$ is equipped with this distance. (why? try proving it yourself)

this distance.

A Banach space is a normed space $(X, \|\cdot\|)$ that is complete, i.e., every Cauchy seq in X converges.

~~Optional~~ note (cont')

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are comparable if $\exists a, b > 0$ s.t.

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \quad \forall x \in X.$$

If this is the case, the metrics they induce are comparable and hence $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$ have the same topology.

When are two Banach spaces homeomorphic? The following almost completely answers this question.

Fact. (Anderson - Kader thm) (there is a countable subset whose closure is the entire space)

Any two infinite dimensional separable Banach spaces are homeomorphic. In fact, they are homeomorphic to $\mathbb{R}^\omega$, the countable product of $\mathbb{R}$, with the product topology.

(We will not prove this thm in this course.)

One deduces from the above Fact that $(\mathbb{R}^\omega, \text{product})$ is metrizable.

We can prove this directly.

Thm. Let (X_n, d_n) be a sequence of metric spaces, $n \in \mathbb{N}$.

Then the product $\prod_{n=1}^{\infty} X_n$ w/ the product topology is metrizable.

pf) If we define $\bar{d}_n : \cancel{X_n \times X_n} \rightarrow \mathbb{R}$ by

$$\bar{d}_n(x, y) = \min\{d_n(x, y), 1\}, \quad x, y \in X_n,$$

then $\bar{d}_n$ induces the same topology as d_n on X_n . The following

$$d : \prod_{n=1}^{\infty} X_n \times \prod_{n=1}^{\infty} X_n \rightarrow \mathbb{R},$$

$$d\left((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}\right) = \sup_{n \geq 1} \frac{1}{n} d_n(x_n, y_n)$$

defines a distance on $\prod_{n=1}^{\infty} X_n$ that induces the product topology. $\square$

However, if J is uncountable, then $(\mathbb{R}^J, \text{product})$ is not metrizable. We will prove this shortly.

Defn Given a metric space (Y, d) and an index set J , the map $d_{\text{uniform}}: Y^J \times Y^J \rightarrow \mathbb{R}$ defined by

$$d_{\text{uniform}}(\{x_j\}_{j \in J}, \{y_j\}_{j \in J}) = \sup_{j \in J} \min\{d(x_j, y_j), 1\}$$

is a metric on Y^J , called the uniform metric on Y^J .

This induces the uniform topology on Y^J .

Thm On Y^J , the box topology is finer than the uniform topology, which in turn is finer than the product topology.

Rmk If J is infinite and $Y = \mathbb{R}$, these three topologies are different.

§21. Additional properties of the metric topology.

~~Subspaces of metric spaces~~

- (X, d) metric space, $A \subseteq X$, define restriction $d_A: A \times A \rightarrow \mathbb{R}$ as $d_A = d|_{A \times A}$. Then topology of (A, d_A) is the subspace topology inherited from (X, d) .
- metric spaces are Hausdorff.
- Def. A top'l sp is first countable if ^{for} each point $x \in X$ there is a countable basis at x , i.e. a countable collection $\{U_n\}_{n=1}^{\infty}$ of open nbhds of x s.t. every nbhd of x contains some U_n .

Then metric spaces (X, d) are first countable, e.g. take $U_n = B(x, \frac{1}{n})$.

Sequence Lemma

Let X be a top'l space and $A \subseteq X$. If there is a sequence of points of A converging to $x \in X$, then $x \in \bar{A}$. The converse holds if X is first countable.

Thm Let $f: X \rightarrow Y$, X, Y top'l sp.

If f is conti and $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$.

Conversely, if X first countable and $f(x_n) \rightarrow f(x)$ whenever $x_n \rightarrow x$, then f conti.

pf) $\Leftarrow$ Enough to show $f(\bar{A}) \subseteq \overline{f(A)}$ for $A \subseteq X$.

But $x \in \bar{A} \Rightarrow \exists x_n \in A, x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x) \Rightarrow f(x) \in \overline{f(A)}$. $\square$

• ϵ - δ condition:

Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $f: X \rightarrow Y$ be a function.

Then f is conti iff $\forall x \in X \forall \epsilon > 0 \exists \delta > 0 d_X(x, y) < \delta \rightarrow d_Y(f(x), f(y)) < \epsilon$.

• Defn. Let X be a set and (Y, d) be a metric space.

A sequence of functions $f_n: X \rightarrow Y$ converges uniformly to $f: X \rightarrow Y$

iff $\forall \epsilon > 0 \exists N \forall n \geq N \forall x \in X d(f_n(x), f(x)) < \epsilon$.

Exercise. $(\Leftrightarrow)$ as elements of Y^X , $f_n \rightarrow f$ as $n \rightarrow \infty$ w.r.t. the uniform topology on Y^X .

Exercise $f_n \rightarrow f$ pointwise as $n \rightarrow \infty \Leftrightarrow f_n \rightarrow f$ as $n \rightarrow \infty$ in Y^X w/ the product topology.

Uniform Limit Thm

X topol sp, Y metric space, $f_n: X \rightarrow Y$ conti, $f: X \rightarrow Y$ map,

f_n converges uniformly to $f \Rightarrow f$ conti.

Exercise. $(\Leftrightarrow)$ The space $C(X, Y)$ of conti fns $X \rightarrow Y$ is closed in $(Y^X, \text{uniform topology})$.

• $(\mathbb{R}^\omega, \text{box topology})$ is not metrizable.

pf) Let $A = (0, \infty)^\omega$. Then $\vec{0} = (0, 0, \dots) \in \bar{A}$.

However, no sequence $\left((x_{nm})_{m=1}^\infty \right)_{n=1}^\infty$ of A can converge to $\vec{0}$, because of the nbhd $\prod_{n=1}^\infty (-x_{nn}, x_{nn})$ of $\vec{0}$.

• $(\mathbb{R}^J, \text{product topology})$, J uncountable, is not metrizable.

pf) Let $A = \{ (x_j)_{j \in J} : x_j = 1 \text{ for all but finitely many } j \}$.

Let $\vec{0} = (0)_{j \in J}$. Then $\vec{0} \in \bar{A}$.

But no sequence $(x_n)_{n=1}^\infty$ of A can converge to $\vec{0}$.

If it did, choose $j \in J \setminus \bigcup_{n=1}^\infty \{ j' \in J : (x_n)_{j'} \neq 1 \}$ ($\neq \emptyset$),

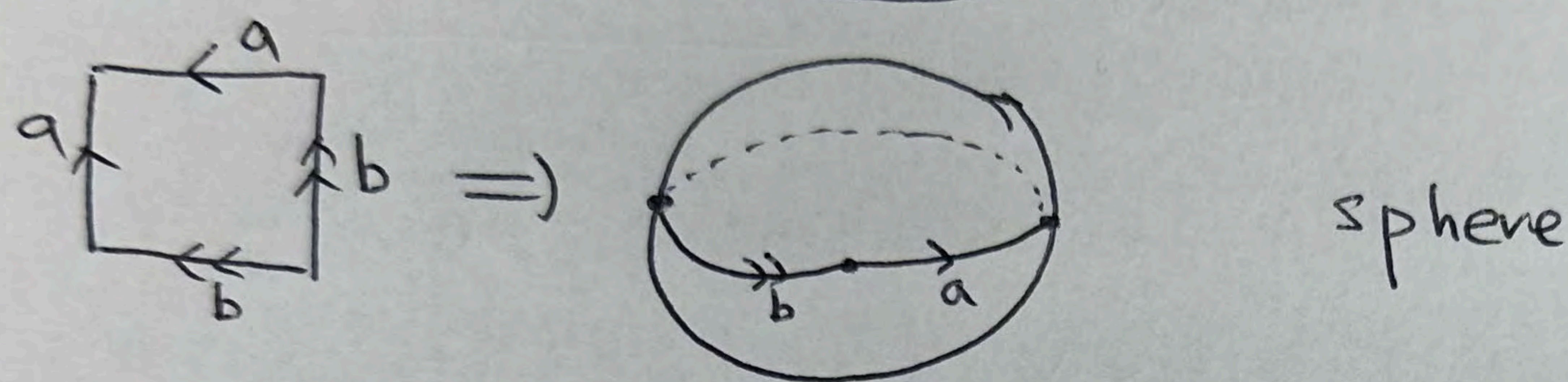
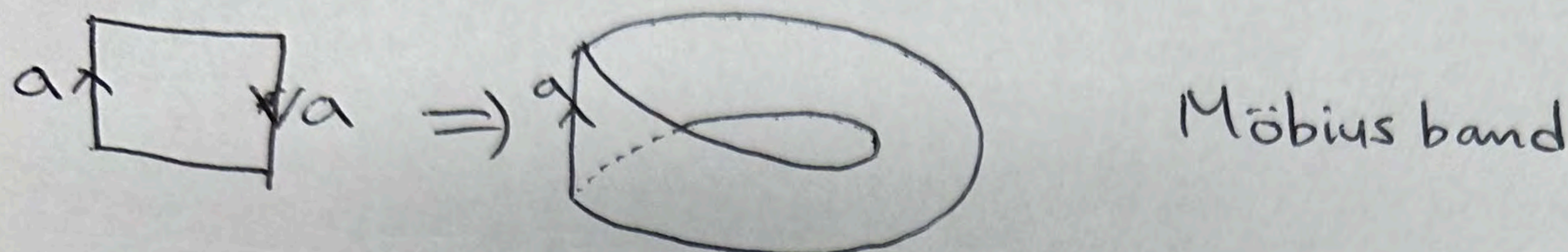
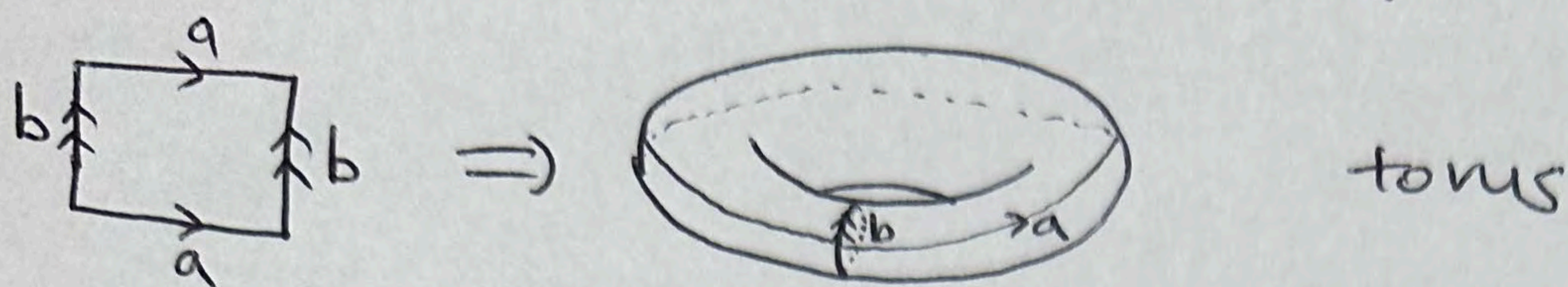
so that $(x_n)_j = 1$ for all n . The map $\pi_j: \mathbb{R}^J \rightarrow \mathbb{R}$ is continuous,

but

$$\pi_j(x_n) = 1 \not\rightarrow 0 = \pi_j(\vec{0}).$$

§22 The Quotient Topology

We wish to formalize the notion of "gluing":



Def. A surjective map $p: X \rightarrow Y$ between topol spaces is a quotient map if

$U \subseteq Y$ open iff $p^{-1}(U) \subseteq X$ open.

$\Leftrightarrow F \subseteq Y$ closed iff $p^{-1}(F) \subseteq X$ closed

$\Leftrightarrow p$ conti & maps saturated open sets of X to open sets of Y .
 (means that it is of the form)
 $p^{-1}(A)$ for some $A \subseteq Y$

ex. A open or closed, surjective conti map is a quotient map.

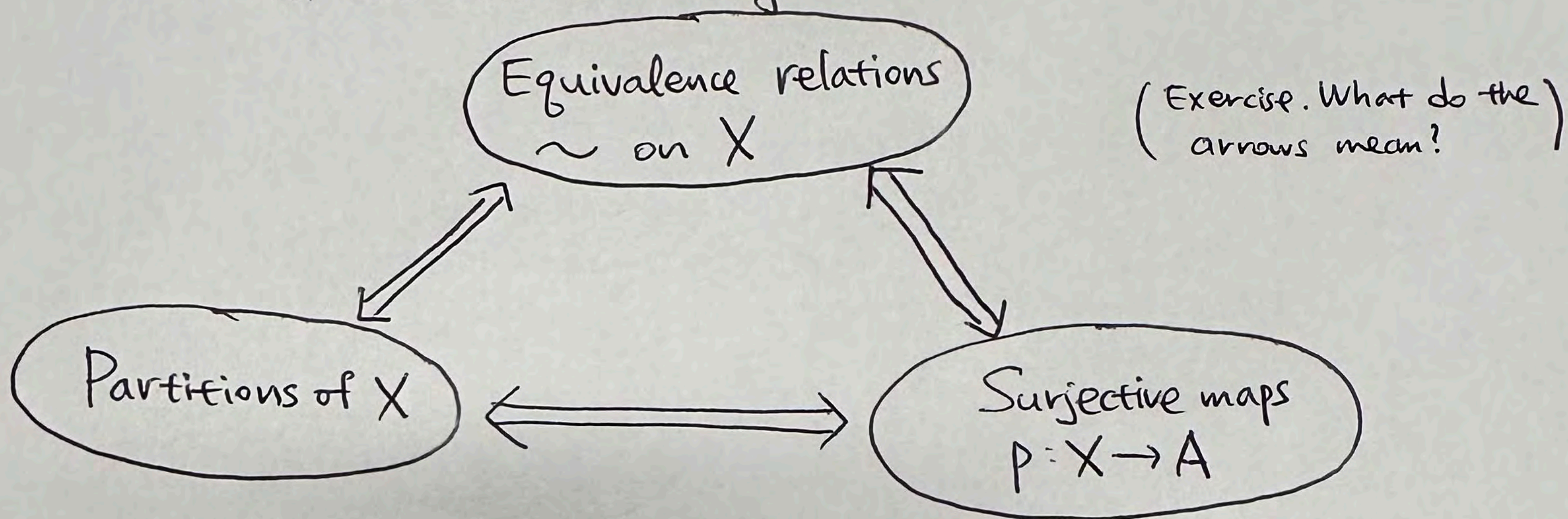
Def. X topol space, A set, $p: X \rightarrow A$ surjection.

Thm. There is exactly one topology on A st. p is a quotient map.

This topology is called the quotient topology induced by p .

Check Of course, if X, Y topol spaces and $p: X \rightarrow Y$ quotient map, then topology of Y is the quotient topology induced by p .

Let X be a set. We have a trinity:



Whenever you define some concept on one of these, you can define an analog of that concept on the others. For example:

Def. Let X top'l sp, X^* a partition of X . Let $p: X \rightarrow X^*$ be the natural surj map.
 (i.e. X^* is a collection of nonempty disjoint subsets of X whose union is X .
 the map $p: X \rightarrow X^*$ sends $x \in X$ to the elt of X^* containing x .)

Give X^* the quotient topology induced by p .

X^* is called ~~the~~ quotient space / identification space / decomposition space of X .
 (disj)

Rmk. An open set of X^* is a collection of $\underset{\text{(closed)}}{\text{subsets}}$ whose union is open in X .
 (closed)

Thm. X^* is T_1 iff each elt of X^* is closed in X .

It is harder to check if X^* is Hausdorff.

• Whenever you define a new concept, you want to be able to mass-produce examples. There aren't many for quotient maps.

Thm. Let $p: X \rightarrow Y$ be a quotient map and $A = p^{-1}(B)$ for $B \subseteq Y$. ("A is saturated")

If A is open or closed or if p is open or closed, then $p|_A: A \rightarrow B$ is a quotient map.

pf) If A open, then B open, and $U \subseteq B \text{ open} \Leftrightarrow U \subseteq Y \text{ open} \Leftrightarrow p^{-1}(u) \subseteq X \text{ open} \Leftrightarrow p^{-1}(u) \in A \text{ open.}$

Similarly if A closed.

If p open, then $p^{-1}(u) \subseteq A \text{ open} \Rightarrow p^{-1}(u) = A \cap O, O \subseteq X \text{ open} \Rightarrow U = B \cap p(O), p(O) \subseteq Y \text{ open} \Rightarrow U \subseteq B \text{ open. Reverse by conti of } p. \text{ Similar if } p \text{ closed.}$

Thm. $p: X \rightarrow Y, q: Y \rightarrow Z$ quotient maps $\Rightarrow q \circ p: X \rightarrow Z$ quotient map.

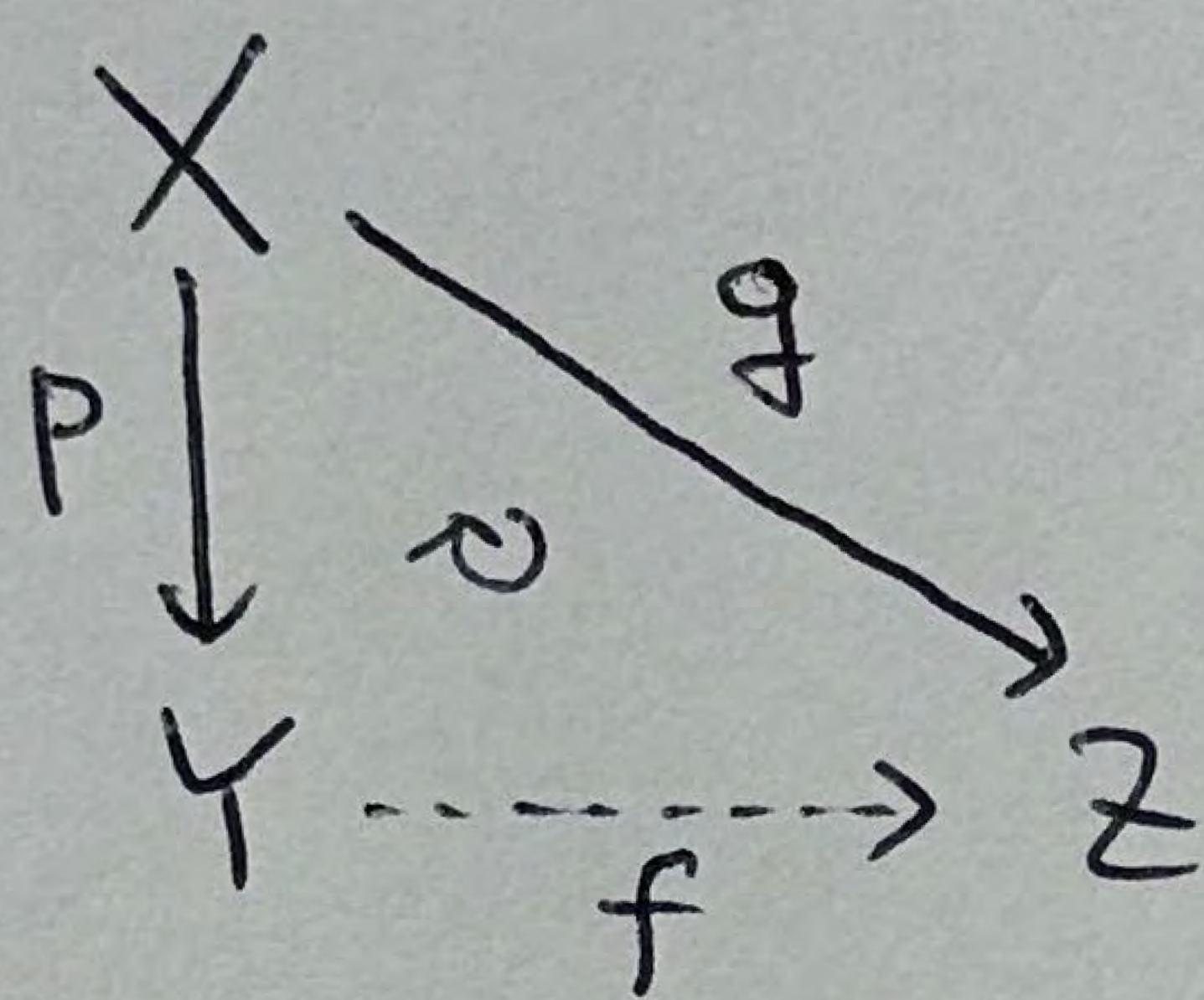
Thm (Optional, revisit after Ch.3)

X "cpt", Y Hdf, $p: X \rightarrow Y$ surj $\Rightarrow p$ quotient.

$p: A \rightarrow B, q: C \rightarrow D$ quotient maps, $\Rightarrow p \times q: A \times C \rightarrow B \times D$ quotient.
 B, C "locally compact" Hausdorff

"Universal Property"

Thm. Let $p: X \rightarrow Y$ be a quotient map, $g: X \rightarrow Z$ a map s.t.
 g const on each set $p^{-1}(\{y\})$ ("fibre of p "). Then g induces
a map $f: Y \rightarrow Z$ s.t. $f \circ p = g$.



Also, f conti iff g conti, f quotient iff g quotient.

Given a ^{surj} conti map $g: X \rightarrow Z$, the following elucidates the relation between the
identification space X^* induced by g and the topology of Z . (note that
 X^* and Z are in 1-1 correspondence.)

Cor. Let $g: X \rightarrow Z$ surj conti map, $X^* = \{g^{-1}(z) : z \in Z\}$ w/ quotient top.

(a) The induced map $f: X^* \rightarrow Z$ s.t. $f \circ p = g$ is bij conti.

f is a homeo iff g quotient.

(b) If Z is Hausdorff, so is X^* . $\leftarrow$ (Exercise. If $f: Y \rightarrow Z$ injective conti,
 Z Hdf, then Y Hdf.)

(c) If X "compact", Z Hausdorff, then X^* and Z are homeo.
(Optional)

§ Topological Groups.

Def A topological group G is a group w/ a T_1 topology s.t. the binary operation $\cdot: G \times G \rightarrow G, (g, h) \mapsto gh$ and the inverse $\cdot^{-1}: G \rightarrow G, g \mapsto g^{-1}$ are continuous.

($\Leftarrow$) the map $G \times G \rightarrow G, (g, h) \mapsto gh^{-1}$ is conti.)

Examples

- $(\mathbb{Z}, +)$
- $(\mathbb{R}, +)$
- $(\mathbb{R}_{>0}, \text{multiplication})$
- $(S^1 = \{z \in \mathbb{C} : |z| = 1\}, \text{multiplication})$
- $GL(n, \mathbb{R}) = \{\text{invertible } n \times n \text{ real matrices}\}, GL(n, \mathbb{C}).$

Thm. G top'l gp, $H \leq G$ subgp $\Rightarrow H, H$ top'l gps.

Ex Subgps of $GL(n)$ as a top'l gps are called matrix groups.

Def/Thm Let G be a top'l gp, $H \leq G$ subgp. A left coset of H is a set of the form $xH = \{xh \mid h \in H\}$, let $G/H = \{xH : x \in G\}$, which is a partition of G . Give G/H the quotient topology induced by the map $p: G \rightarrow G/H, p(x) = xH$. Then p is open. If H is closed, then G/H is T_1 . If H is closed and normal, then G/H is a topological gp.

Ex $G = (\mathbb{R}, +), H = \mathbb{Z} \Rightarrow \mathbb{R}/\mathbb{Z} = S^1$.

Thm A top'l gp G is Hausdorff.

pf) Let $x, y \in G, x \neq y$. Then $G \setminus \{x^{-1}y\}$ is an open nbhd of the identity e_G , so $\exists U, V \ni e_G$ open s.t. $UV \subseteq G \setminus \{x^{-1}y\}$. Then $xU \cap yV^{-1} = \emptyset$.
by conti of binary op,

§23 Connected Spaces.

Def. Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X :

$$U \cap V = \emptyset, \quad U \cup V = X.$$

$\Leftrightarrow$ A pair A, B of subsets of X s.t.

$$\bar{A} \cap B = \emptyset, \quad A \cap \bar{B} = \emptyset, \quad A \cup B = X$$

$\Leftrightarrow$ A and $X \setminus A$ for a closed and ~~closed~~ open subset A of X that is nonempty and $A \neq X$.

X is connected iff there is no separation of X . (this is a topol property.)
iff $\nexists$ "clopen" $A \subseteq X$ s.t. $A \neq \emptyset, X$.

Lemma. If C, D is a separation of X and Y is a connected subsp of X , then $Y \subseteq C$ or $Y \subseteq D$.

Some ways to make new connected spaces:

Thm. If $A_\alpha \subseteq X$, A_α connected, $\bigcap_\alpha A_\alpha \neq \emptyset$, then $\bigcup_\alpha A_\alpha$ connected.

Thm. If $A \subseteq B \subseteq \bar{A} \subseteq X$, A connected, then B connected.

Thm. If $f: X \rightarrow Y$ conti, X connected, then $f(X)$ connected.

Thm. X_α connected $\Rightarrow \prod_\alpha X_\alpha$ w/ product topology connected.

Thm. $p: X \rightarrow Y$ conti, quotient, each fibre $p^{-1}\{y\}$ connected, Y connected $\Rightarrow X$ connected.

Ex. $\mathbb{R}^n$ w/ box topology is not connected.

$\therefore \{ \text{bdd segs} \} \cup \{ \text{unbdd segs} \}.$

Def. A space X is totally disconnected if its only connected subsp are singletons.

Ex. Let p be a prime. Consider p -adic integers which are formal power series $a = a_0 + a_1 p + a_2 p^2 + \dots$, $a_i \in \{0, \dots, p-1\}$, $\mathbb{Z}_p = \{p\text{-adic integers}\}.$

Define p -adic norm $|a|_p = p^{-k}$ where k is smallest s.t. $a_k \neq 0$. ($|0|_p = 0$).
Then $d_p(x, y) = |x - y|_p$ is an ultrametric: $|x + y|_p \leq \max\{|x|_p, |y|_p\}.$

Exercise: $(\mathbb{Z}_p, d_p)$ is totally disconnected.

§24. Connected Subspaces of $\mathbb{R}$.

Def. A simply ordered set L (w/ ≥ 2 elts) is a linear continuum if

- (1) L has the least upper bound property:
every nonempty subset A_0 of A that is bdd above ($\exists b \in A \forall a \in A_0 a \leq b$) has a least upper bound (an upper bound b_0 s.t. for every upper bound b of A_0 , $b_0 \leq b$).
- (2) If $x < y$, $\exists z$ s.t. $x < z < y$.

Ex. $\mathbb{R}$ and intervals thereof. Ordered square I_0^2 .
 $X \times (0,1)$ w/ dictionary order where X well-ordered.

Thm. If L is a linear continuum w/ order topology,
then intervals, rays ~~and~~ of L are connected.

pf) ~~S'pse $\forall x \in L$ is convex (i.e. $a < x < b \Rightarrow [a,b] \subseteq X$).~~

S'pse $a, b \in L$, $a < b$. Enough to show $[a,b]$ connected.

S'pse $[a,b] = U \cup V$, $a \in U$. Let

$$\mathcal{I} = \{c \in L : [a,c] \subseteq U\}.$$

Then $\mathcal{I} \neq \emptyset$ since U open and property (2) of L . Let $b_0 = \sup \mathcal{I}$.

Then $b_0 = b$ since U open and property (2) of L . So $[a,b) \subseteq U$.

If $b \in V$, then $U \cap V \neq \emptyset$ since V open and prop (2). So $[a,b] \subseteq U$.

Cor. Intervals and rays of L are connected.

Intermediate Value thm

X conn, Y order topology, $f: X \rightarrow Y$ conti, a

$a, b \in X$, $r \in Y$ between $f(a), f(b) \Rightarrow \exists c \in X$ $f(c) = r$.

Def. (Path-connectedness).

A path in a topol sp X from $x \in X$ to $y \in X$ is a conti map $f: [0,1] \rightarrow X$
s.t. $f(0) = x$, $f(1) = y$. X is path-connected if every pair of pts can be
joined by a path.

A path connected space is connected.

Ex Unit ball $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ of $\mathbb{R}^n$ (and of any normed space).

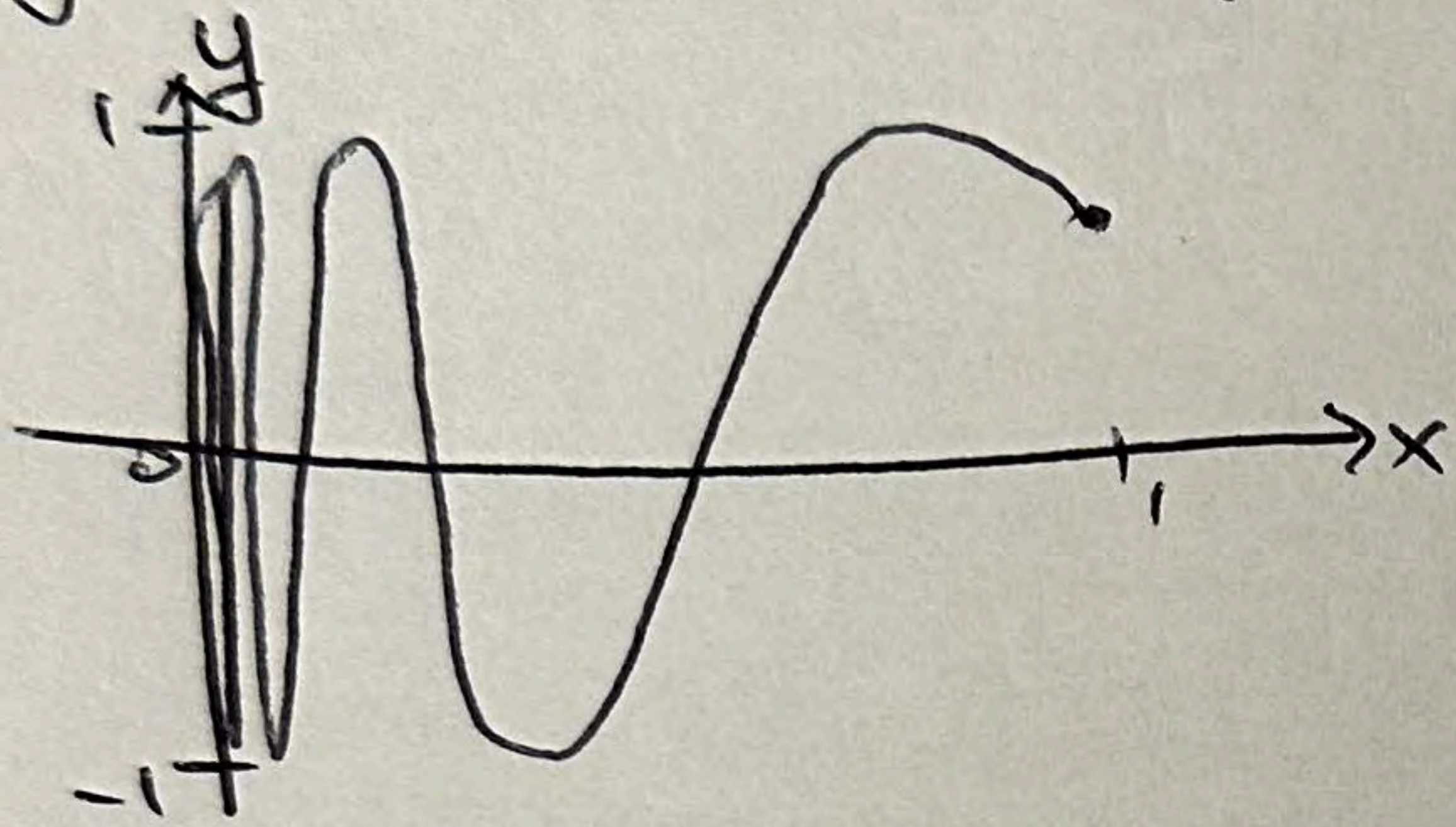
Ex Punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$.

($\Rightarrow \mathbb{R}, \mathbb{R}^n (n \geq 2)$ not homeomorphic!)

Ex Unit sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ of $\mathbb{R}^n$ (and of any normed space) of $\dim \geq 2$
 S^{n-1} $n \geq 2$

$\therefore$ Conti map $\mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$, $x \mapsto x/\|x\|$.

Ex Topologist's sine curve $\{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}$.



"Smallest" uncountable well-ordered set.



Ex, I_0^2 is conn. but not path-connected. Same for long line $S_{\aleph_1} \times [0, 1]$.

§25. Components and local connectedness

Def. Top'l sp X , $x, y \in X$.

$x \sim_c y$ if there is a connected subsp $C \ni x, y$.

$\Rightarrow$ equiv classes called (connected) components of X .

$x \sim_p y$ if there is a path from x to y

$\Rightarrow$ equiv classes called path components of X .

Thm. Components of X are connected disj. subsp of X whose union is X .

Each nonempty connected subsp of X intersects only one.

• Path components of X are path-conn. disj. subsp. of X whose union is X .

Each nonempty path-conn. subsp of X intersects only one.

• Each path component lies in a component.

ex For I_0^2 , comp = I_0^2 , path comp = vertical lines.

When do components and path components coincide?

Def. X is locally (path) connected at $x \in X$ if $\forall$ nbhd U of x ,
 $\exists$ (path) connected nbhd V of x , $V \subseteq U$.

X is locally (path) connected if so at each $x \in X$.

Thm X is locally (path) connected iff $\forall$ open $U \subseteq X$, each (path) comp of U open in X .

Thm If X locally path connected, components = path comps.

ex. Open subsets of $\mathbb{R}^n$ are loc. path con.

§10. Well-ordered sets.

Def A simply ordered set A is well-ordered if every nonempty subset of A has a smallest elt.

Constructing well-orderings.

(1) Finite simply ordered sets are well-ordered.

(2) Subsets of "wosets" are wosets.

(3) Finite products of "wosets" w/ dictionary order are wosets.

(4) (Zermelo 1904) (Well-ordering thm).

Any set A has a well-ordering.

Cor. $\exists$ uncountable woset.

Def. Given woset X , $\alpha \in X$, define section
 $S_\alpha = \{x \in X : x < \alpha\}$.

Thm. $\exists$ woset A w/ largest elt Ω s.t.

(1) S_Ω uncountable.

(2) $\alpha < \Omega \rightarrow S_\alpha$ countable.

(3) $A' \subseteq S_\Omega$ countable $\rightarrow A'$ has an upper bd.

§26. Compact spaces.

Def. Let $\mathcal{A}$ be a collection of subsets of X . Let $Y \subseteq X$.

$\mathcal{A}$ covers Y if $Y \subseteq \bigcup \mathcal{A}$.

$\mathcal{A}$ is an open cover if each $A \in \mathcal{A}$ is open (when X is a top'l sp.)

$\mathcal{A}$ is a finite cover if $\mathcal{A}$ contains only finitely many sets.

Def. A top'l sp X is compact if every open cover of X has a finite subcover.

ex. Finite top'l spaces are compact.

$\mathbb{R}, \mathbb{Z}$ are not compact.

Lemma. (Compactness = Relative compactness)

$Y \subseteq X$ top'l sp. Y cpt $\Leftrightarrow$ every covering of Y by open sets in X has a finite subcollection covering Y .

pf) Defn of subsp top. $\square$

Thm. A closed subsp Y of a compact space X is cpt.

pf) To an open cover of Y , adjoin $\{X \setminus Y\}$. $\square$

Thm. A cpt subsp Y of a Hdf space X is closed.

pf) For each $x \in X \setminus Y$, for each $y \in Y$ choose disjoint $x \in U_y$, $y \in V_y$.

Then $\{U_y : y \in Y\}$ is an open cover of Y w/ finite subcovr $V_{y_1}, \dots, V_{y_n}$.

Then

$$x \in U_{y_1} \cap \dots \cap U_{y_n}, Y \subseteq V_{y_1} \cup \dots \cup V_{y_n}, (U_{y_1} \cap \dots \cap U_{y_n}) \cap (V_{y_1} \cup \dots \cup V_{y_n}) = \emptyset. \quad \square$$

Thm. $f: X \rightarrow Y$ conti, X cpt $\Rightarrow f(X)$ cpt.

pf) "Pull-back" an open cover of $f(X)$ to an open covr of X .

Thm. $f: X \rightarrow Y$ bij conti, X cpt Y Hdf $\Rightarrow f$ homeo.

pf) $C \subseteq X$ clsd $\rightarrow C$ cpt $\rightarrow f(C) \subseteq Y$ cpt $\rightarrow f(C) \subseteq Y$ clsd.

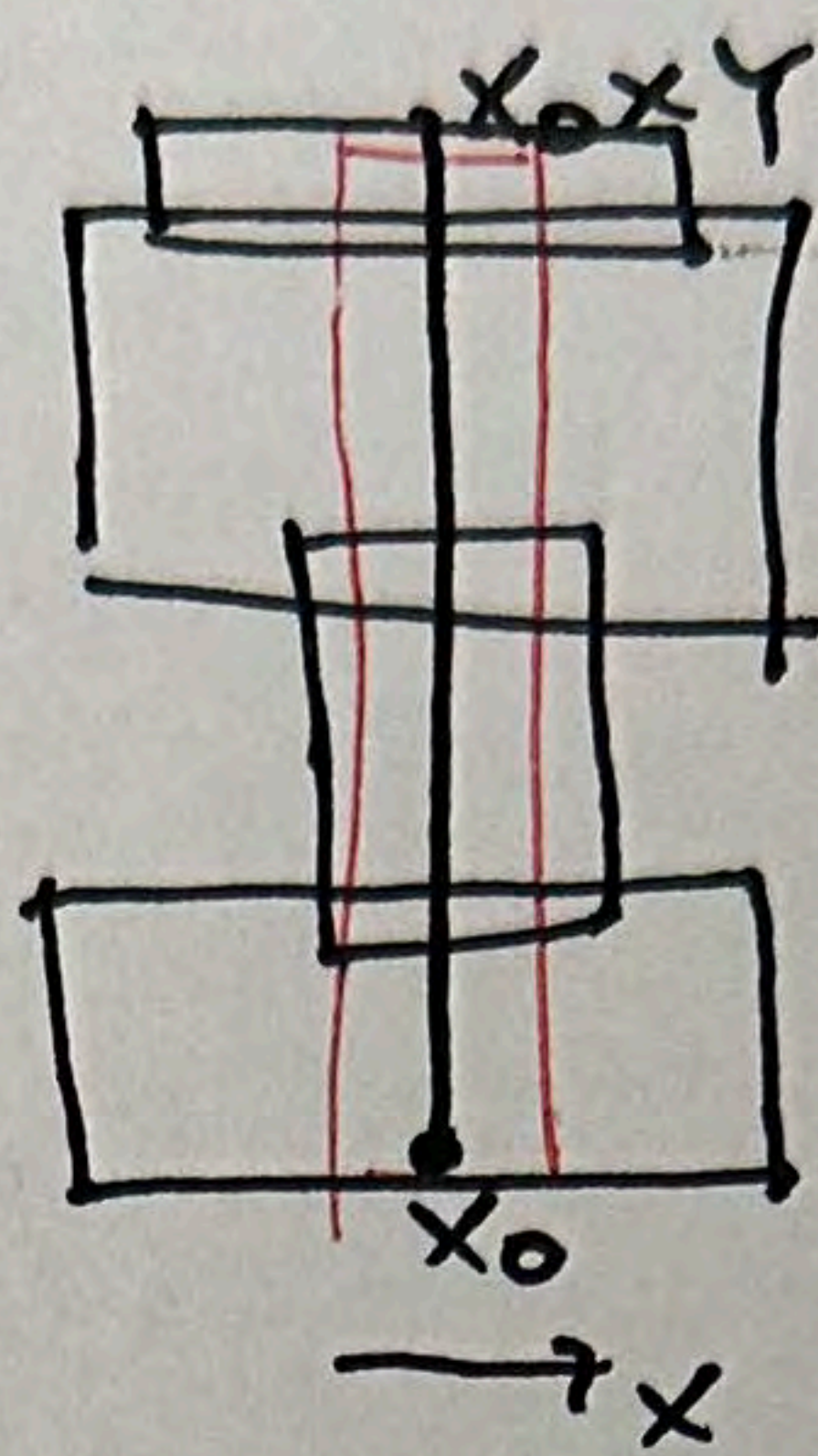
Thm. X, Y cpt $\Rightarrow X \times Y$ cpt.

pf) Tube lemma: X top'l sp, Y cpt, $x_0 \in X$, U open nbhd of $x_0 \times Y$
 $\Rightarrow$ open set $x_0 \in V$ s.t. $x_0 \times Y \subseteq V \times Y \subseteq U$.

pf) $\forall y \in Y \exists x_0 \in V_y$ open $\exists y \in W_y$ open $(x_0, y) \in V_y \times W_y \subseteq U$.

Then ~~$\{V_y: y \in Y\}$ open cov of X~~ $\{W_y: y \in Y\}$ open cov of Y w/ finite subcover.

Take $V = V_{y_1} \cap \dots \cap V_{y_n}$. $\square$
 $\uparrow$
 Y



Let A be an open cov of $X \times Y$. For each $x_0 \in X$, similar to tube lemma, may ~~cover~~ find finite cover of some $U_{x_0} \times Y$, $x_0 \in U_{x_0}$ open.
 But $\{U_{x_0}: x_0 \in X\}$ has finite subcover of X . $\square$

Another formulation of cptness in terms of closed sets.

Def. A collection $\mathcal{C}$ of subsets of X has the finite intersection property if $\forall C_1, \dots, C_n \in \mathcal{C}$, $C_1 \cap \dots \cap C_n \neq \emptyset$.

Thm A top'l sp X is cpt iff any ^(nonempty) collection $\mathcal{C}$ of closed sets of X w/ the finite intersection property satisfies $\bigcap \mathcal{C} \neq \emptyset$.

Cor. If X is cpt, then any nested sequence

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots \supseteq C_n \supseteq \dots$$

of nonempty closed sets have nonempty intersection $\bigcap C_n \neq \emptyset$.

§27. Compact subspaces of the real line.

Thm. $[0,1]$ is cpt.

pf) Let $\mathcal{A}$ be an open cover, let

$$\mathcal{C} = \{x \in [0,1] : [0,x] \text{ covered by finitely many elts of } \mathcal{A}\}.$$

Then $\mathcal{C}$ nonempty, so $\exists x_0 = \sup \mathcal{C} \in [0,1]$. Sup. is attained and $x_0 = 1$. $\square$

Thm. Closed intervals in a simply ordered set w/ the least upper bound prop. are compact.

Thm. $A \subseteq \mathbb{R}^n$ cpt iff closed and bounded.

pf) $\Rightarrow$] cpt subset of $\mathbb{H}^d$ is clsd; $\bigcup_m B_m$ is an open cov of A .

$\Leftarrow$] A is closed subsp of $[-M,M]^n$. $\square$

Extreme value thm.

$f: X \rightarrow Y$ conti, X cpt, Y order topology $\Rightarrow f$ attains max/min.

pf) $f(X)$ cpt, so enough to show that $K \subseteq Y$ cpt has max elt.

If it doesn't, $\{(-\infty, k) : k \in K\}$ is an open cov of K . $\square$

Lebesgue number lemma.

(X,d) compact metric space, $\mathcal{A}$ open covering of X .

$$\Rightarrow \exists \delta > 0 \quad \forall B \subseteq X \quad \text{diam } B < \delta \rightarrow \exists A \in \mathcal{A} \quad B \subseteq A.$$

pf) take finite subcover $A_1, \dots, A_n$ of X . Define $f: X \rightarrow \mathbb{R}$,

$$f(x) = \max_{i=1, \dots, n} d(x, X \setminus A_i) > 0 \quad \forall x \in X.$$

Say $\min_{x \in X} f(x) = \delta$. $\square$

Uniform continuity thm.

Let X cpt m.s., Y m.s., $f: X \rightarrow Y$ conti. Then f is uniformly continuous:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad d(x_0, x_1) < \delta \rightarrow d(f(x_0), f(x_1)) < \varepsilon.$$

pf) Lebesgue number of covering $f^{-1}(B_\varepsilon(y))$, $y \in Y$.

§28. Limit point compactness.

Def. X is limit point compact if every infinite subset of X has a limit point.

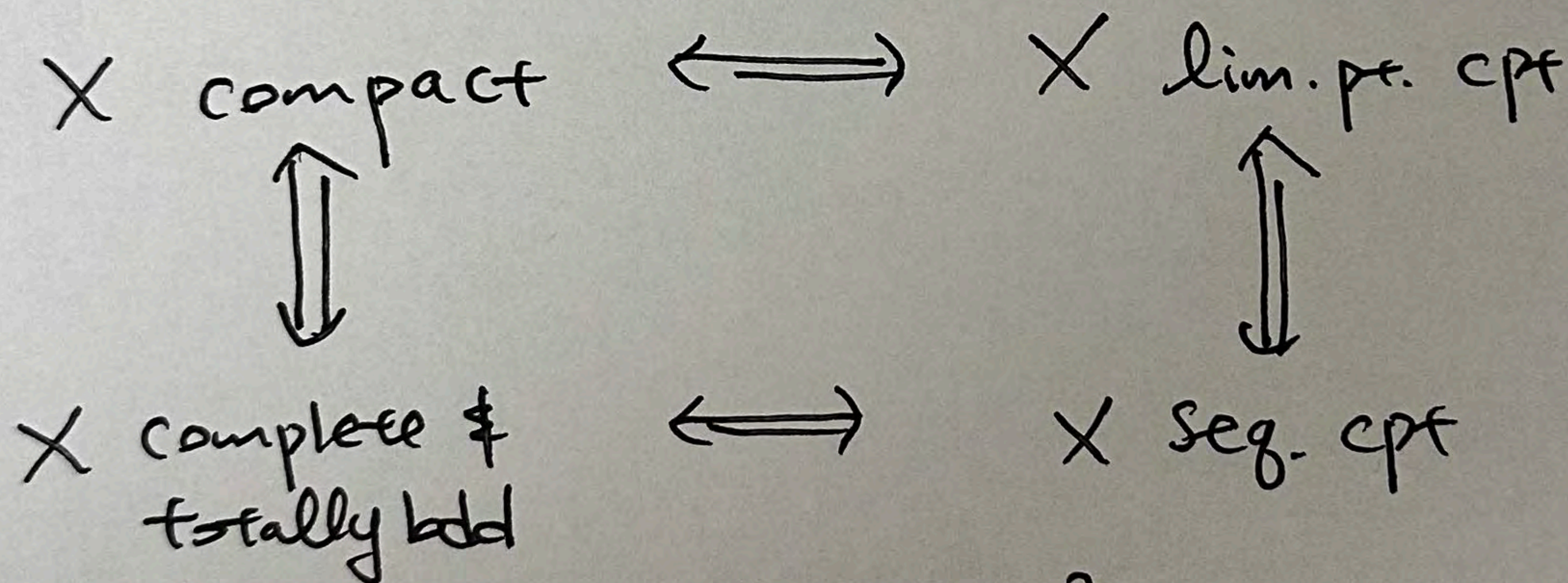
X is sequentially compact if every seq in X has a convergent subsequence.

Thm. Compact $\Rightarrow$ limit point compact $\Rightarrow$ sequentially compact.
 pt) If A w/o limit pts, then each $a \in A$ isolated, i.e. $\exists U_a$ open s.t. $U_a \cap A = \{a\}$. $\{U_a : a \in A\}$ cover of A .
 If $\{x_n : n \in \mathbb{N}\}$ finite, it repeats a value infinitely. Otherwise $\{x_n : n \in \mathbb{N}\}$ infinite. $\square$

Def ~~Rmk~~. A metric space (X, d) is totally bounded if $\forall \varepsilon > 0 \exists x_1, \dots, x_n \in X$
 $X = B(x_1, \varepsilon) \cup \dots \cup B(x_n, \varepsilon)$.

A metric space (X, d) is complete if every Cauchy seq. converges.

Thm Let X be a metric space.



pt) $[X \text{ seq. cpt} \Rightarrow X \text{ complete \& tot. bdd}]$

If $\{x_n\}$ is a Cauchy seq, it has a subseq $x_{n_k} \rightarrow x$. Then $x_n \rightarrow x$.

If X not totally bdd, then inductively choose $x_1, x_2, \dots$ s.t. $d(x_i, x_j) > \varepsilon$ for $i \neq j$.

This cannot have a conv. subseq. $\star$

$[X \text{ compl. tot bdd} \Rightarrow X \text{ cpt}]$

S'pse $\mathcal{A}$ open cover of X w/o finite subcover. Take $X = \bigcup_i B(x_i^1, 1)$.

Then some $F_1 = \overline{B(x_i^1, 1)}$ not finite coverable by $\mathcal{A}$. Take $X = \bigcup_i B(x_i^2, 1/2)$

so $F_1 = \bigcup_i F_1 \cap \overline{B(x_i^2, 1/2)}$. Some $F_2 = F_1 \cap \overline{B(x_i^2, 1/2)}$ not finite coverable by $\mathcal{A}$.

Continue inductively, say $\bigcap_n F_n = \{x_0\}$ by completeness. Choose $x_0 \in A \in \mathcal{A}$.

Then $F_n \subseteq A$ for some n . $\star$

§29. Local Compactness.

Def. A topol space X is locally compact at x if $\exists$ nbhd V of x s.t. $\bar{V}$ cpt.

A topol space X is locally compact if locally compact at every $x \in X$.

ex. $\mathbb{R}^n$, simply ordered sets w/ least upper bound property, are locally cpt.
 $\mathbb{R}^{\omega}$ is not.

When is a space locally compact & Hausdorff (LCH)?

Def. If Y is cpt Hdf & $X \subseteq Y$ w/ $\bar{X} = Y$, then Y is a compactification of X .

If, in addition, $Y \setminus X$ is a single pt, Y is a one-point compactification of X .

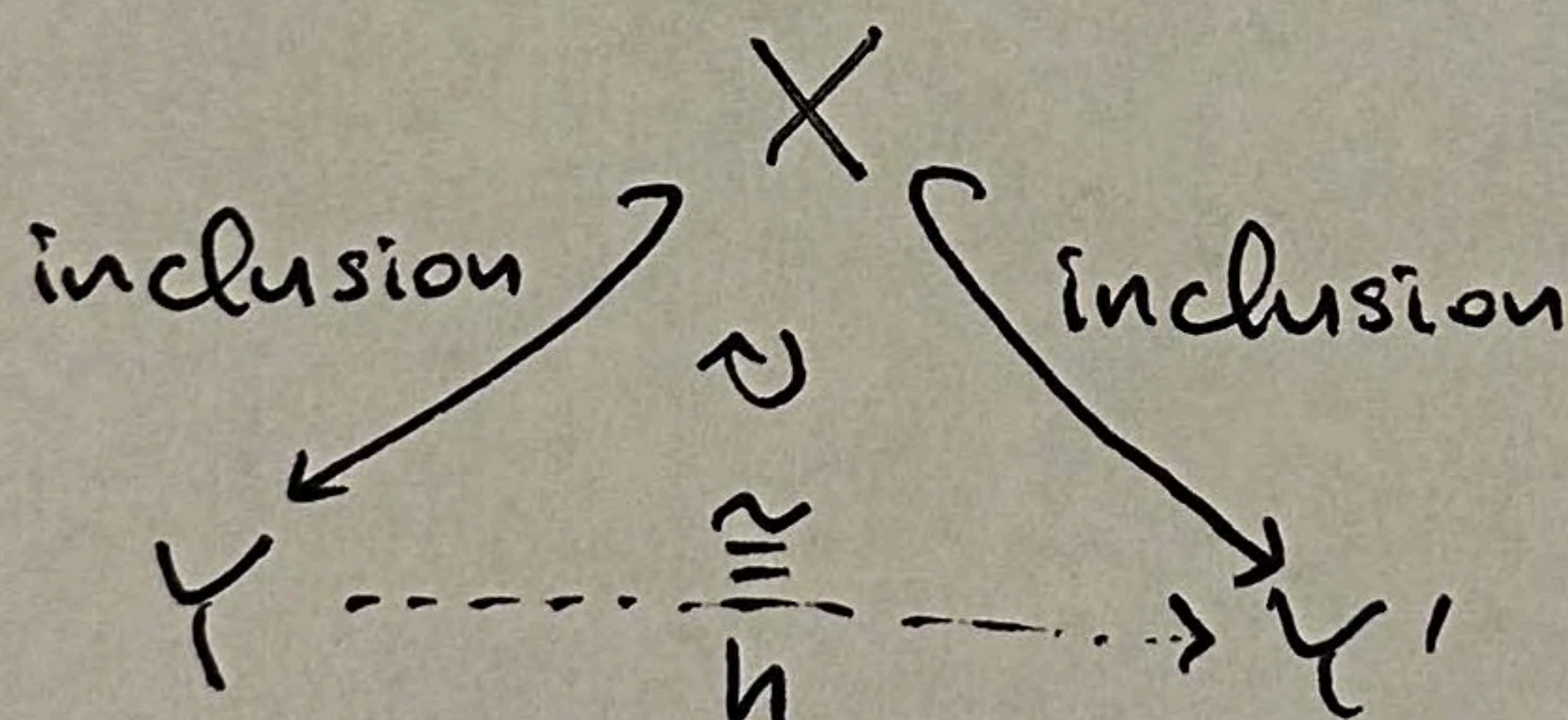
Thm Let X be a topol sp. Then X is LCH iff $\exists$ topol sp Y s.t.

(1) $X \subseteq Y$

(2) $Y \setminus X$ is a single pt $\{y_0\}$

(3) Y is cpt Hdf.

Such a Y is unique, i.e. if Y' were another such space, then there is a homeomorphism $h: Y \rightarrow Y'$ s.t. $h|_X = \text{id}_X$:



pf) $\Leftarrow$ Let $x \in X$. Then $x \neq y_0$, so $\exists$ disj nbhds $x \in U, y_0 \in V, U \cap V = \emptyset$, in Y .

Then $\bar{U} \subseteq Y \setminus V \subseteq X$, U is cpt.

$\Rightarrow$ Define $Y = X \cup \{y_0\}$, define a topology on Y by

$$\{U: U \subseteq X \text{ open}\} \cup \{Y \setminus C: C \subseteq X \text{ cpt}\}.$$

You can verify this is a cpt Hdf topology on Y .

uniqueness] If $Y' = X \cup \{y'_0\}$ were another such space, the map $h: Y \rightarrow Y'$,

$$h(y) = \begin{cases} y & \text{if } y \neq y_0 \\ y'_0 & \text{if } y = y_0 \end{cases} \text{ is a homeo.}$$

ex. If X is cpt Hdf, then $Y = X \cup \{y_0\}$, with $\{y_0\}$ open.

If $X = \mathbb{R}^n$, then $Y \cong S^n$.

If $X = S_{\mathbb{R}}$, then $Y = \bar{S}_{\mathbb{R}}$.

Under Hdf assumption, local cptness is truly local.

Thm. Let X Hdf.

$$X \text{ loc. cpt} \Leftrightarrow \forall x \in U \subseteq X, \exists x \in V \subseteq \bar{V} \subseteq U, V \text{ open}, \bar{V} \text{ cpt.}$$

pf) $\Rightarrow$] May assume $\bar{U}$ cpt. Then $\text{bd } U$ is cpt & disjoint from x ,

so $\exists x \in O_1, \text{bd } U \subseteq O_2$ disj. open nbhds

$$\Rightarrow \text{ ~~} x \in O_1 \cap U, \bar{O}_1 \cap U \subseteq \bar{U} \cap (X \setminus O_2) \subseteq U, \bar{O}_1 \cap U \text{ cpt. } \square \text{.}~~$$

$$x \in O_1 \cap U, \bar{O}_1 \cap U \subseteq \bar{U} \cap (X \setminus O_2) \subseteq U, \bar{O}_1 \cap U \text{ cpt. } \square.$$

Cor. X loc cpt, $A \subseteq X$ closed $\rightarrow A$ loc cpt. pf) by defn.

X LCH, $A \subseteq X$ open $\rightarrow A$ loc cpt. pf) by above thm.

Cor. X is homeo to an open subsp of a cpt Hdf space iff X LCH.

§ Nets.

(2)

We've seen that cptness $\neq$ sequential cptness. There is a generalization of sequences, the convergence condition of which recovers compactness.

Def. A partial order relation on a set A is a relation $\leq$ s.t.

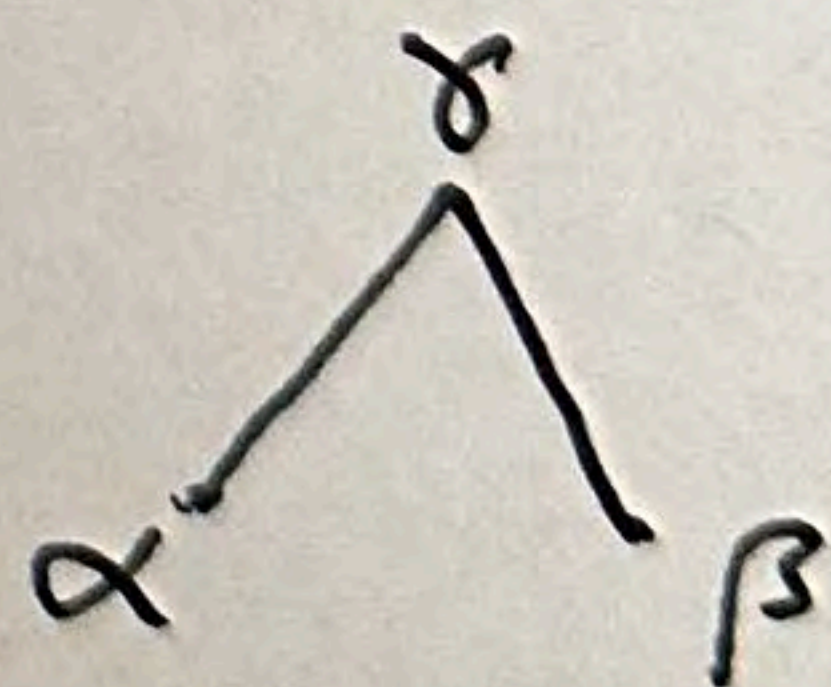
$$(1) \alpha \leq \alpha \quad \forall \alpha.$$

$$(2) \alpha \leq \beta \wedge \beta \leq \alpha \rightarrow \alpha = \beta$$

$$(3) \alpha \leq \beta \wedge \beta \leq \gamma \rightarrow \alpha \leq \gamma.$$

A directed set J is a nonempty set with a partial order $\leq$ s.t.

$$\forall \alpha, \beta \in J \exists \gamma \in J \alpha \leq \gamma \wedge \beta \leq \gamma.$$



ex.. A simply ordered set with its order relation is a directed set.

• The topology $\mathcal{T}$ of a top'l sp X , with $A \leq B$ if $A \supseteq B$, is a directed set.

The set of open nbhds of a pt $x \in X$, _____, _____.

Let's talk about generalization of sequences and convergence.

Def. A net in a topological space X is a function $f: J \rightarrow X$, J a directed set, $f(\alpha) = x_\alpha$, $\alpha \in J$. We write $f = (x_\alpha)_{\alpha \in J} = (x_\alpha)$.

The net (x_α) converges to $x \in X$, or $x_\alpha \rightarrow x$, if $\forall x \in U \text{ nbhd } \exists \alpha \in J$
 $\alpha \leq \beta \rightarrow x_\beta \in U$.

Thm. If $x_\alpha \rightarrow x$ in X and $y_\alpha \rightarrow y$ in Y , then $(x_\alpha, y_\alpha) \rightarrow (x, y)$ in $X \times Y$.

Thm Let $A \subseteq X$. Then $x \in \bar{A}$ iff $\exists$ net of pts of A converging to x .

$\Rightarrow$] Take the directed set J of open nbhds of x .

Define $f: J \rightarrow X$ as $f(\alpha) = (\text{elt of } \alpha \cap A)$ (by Axiom of Choice)

Thm $f: X \rightarrow Y$ conti iff whenever $x_\alpha \rightarrow x$ in X , we have $f(x_\alpha) \rightarrow f(x)$ in Y .

Thm X is Hdf iff every net in X converges to at most one pt.

Let's define subsequences.

Def A subset K of a directed set J is cofinal in J if $\forall \alpha \in J \exists \beta \in K \alpha \leq \beta$.
(automatically K is also a directed set.)

Def If J, K are directed sets, $g: K \rightarrow J$ is a fn s.t.

(i) $i \leq j \rightarrow g(i) \leq g(j)$

(ii) $g(K)$ is cofinal in J ,

and $f: J \rightarrow X$ is a net, then $f \circ g: K \rightarrow X$ is a subnet of f .

~~Def.~~

Def. An accumulation point of the net $(x_\alpha)_{\alpha \in J}$ is a pt x s.t. for any nbhd U of x , $\{\alpha: x_\alpha \in U\}$ is cofinal in J .

$\Leftrightarrow$ some subnet of (x_α) converges to x .

Thm X is cpt iff every net in X has a conv. subnet.

$\Leftrightarrow$] Let $\mathcal{A}$ be a collection of closed sets w/ finite intersection property,
 $J =$ finite intersections of $\mathcal{A}$ ordered by rev. inclusion.